Training of Functional Analysis in the pedagogical higher educational institution: practical course

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Stating the problem. The subject of the course "Functional Analysis" is the scope of functions and their reflection. Functional analysis as an independent section of mathematics developed at the beginning of the last century as a result of the generalization of mathematical analysis structures, linear algebra and geometry. Since then, its ideas and methods have penetrated into all the fields of mathematics, physics and applied sciences on the rights of a powerful generalization theory and a convenient tool for the specific problems study.

The study of functional analysis is typical for mathematical specialties of classical universities. But in the pedagogical higher educational institutions this course is found in the curricula of the specialty 014.04 Secondary education (Mathematics) with the additional specialty 014.09 Secondary education (Computer Science). The bachelor's program in these specialties involves studying the elements of functional analysis. It is usually the basic part of the fundamental cycle with the academic study time one semester, while the number of class hours is small.

The framework of academic study duration, its applied orientation and the students' level of basic training in modern pedagogical universities do not allow them to learn such a complex mathematical discipline from the standpoint of the classical approach, which provides for the fundamental and self-sufficient supply of purely theoretical material. In addition, pragmatically-minded students are not interested in the idea of generalization and formalization of mathematical constructions. Obviously, the motivation increases if you bring the academic course to computing practice with compulsory engagement of computer technology. For future mathematics and computer science teachers it is necessary to emphasize the applied role of functional analysis, which is reduced to the analytical substantiation of the effectiveness of the numerical methods application.

Research and publications analysis. There is only insufficient part of those who put and solve a similar methodological problem among all the published educational resources. For example, the textbooks by V. O. Trenogin, B. M. Pysarevskyi and T. S. Soboleva [1], V. I. Lebedev[2] are oriented on applied specialties. However, they are too voluminous, complex (especially V. O. Trenogin) and, although they cover functional analysis from the point of view of numerical methods, are not completely the path from the idea to the calculation formula, which complicates their use in the pedagogical university. Moreover, essential adaptation of these textbooks is required.

In the problem books of functional analysis it is not accepted to accentuate either the computational or even algorithmic component, and they are not accustomed to actively involve computer technologies. The existing sets are dominated by theoretical tasks. They tend to use purely abstract schemes (space X, norm p, operator A, etc.). The overwhelming majority of such tasks are antipodes of typical calculations. The embodiment of this approach is a problem book by O. O. Kirilov, O. D. Gvishiani[3], recommended for classical universities, however, to this or that degree, all the problem books of functional analysis tend to immerse into the formal logic apparatus. When setting the training problem for future mathematics and computer science teachers, at least, it would be necessary to change the form of the presentation of traditional tasks: to move from abstract to specific spaces, norms, operators, and from tasks to bring to more typical, algorithmic calculations and constructs. This way they are realized in the workshop of Belarus State University[4], though not to the full extend: in the range of tasks, there are no simple enough one-hour exercises aimed at working out the elementary instruments of the discipline.

[1] Треногин, В. А., Писаревский, В. М., & Соболева, Т. С. (1984). Задачи и упражнения по функциональному анализу. Москва: Наука.

[2] Лебедев, В. И. (2005). Функциональный анализ и вычислительная математика (4-е изд.). Москва: Физматлит.

[3] Кириллов, А. А., & Гвишиани, А. Д. (1988). Теоремы и задачи функционального анализа (2-е изд.). Москва: Наука.

[4] Антоневич, А. Б., Ваткина, Е. И., Мазель, М. А., Миротин, А. Р., Мухин, В. В., Радыно, Я. В. et al. (2003). Функциональный анализ и интегральные уравнения: Лабораторный практикум. Минск: БГУ.

Another way of bringing functional analysis to computational practice is to put in the center of the problem the application of any numerical method, followed from the theory. This tendency is revealed episodically in the problem books by V. O. Trenogin, B. M. Pysarevskyi and T. S. Soboleva[5]. Based on the textbook[6], this task allows us to develop the application of functional analysis to numerical methods thoroughly, but, unfortunately, it is carried out mostly from the theoretical side, not practical, and at a high level of abstraction. In addition, the application of numerical methods requires the use of computer technology, and such attempts are found in A. B. Antonevich, Ye. I. Vatkina, M. A. Mazel[7] and V. O. Trenogin[6], but their weight is negligible. One can conclude that in the existing collections functional analysis problem books, there are almost no problems with the application of functional analysis to numerical methods that would be useful for the future mathematics and computer science teacher, and that would allow both mathematical argumentation and implementation with the use of computing.

The objective of the article. To substantiate scientifically the expediency of the developed educational-methodical complex on functional analysis, directed on formation of the future mathematics and computer science teacher's general and professional competences .

Presenting the main material. Insufficient attention to the practice of applying numerical methods in the existing functional analysis manuals compiled for applied specialties can be explained by several circumstances. Firstly, the very ideology of functional analysis is tuned to the high abstractness of this section of mathematics. Secondly, the training trajectories of this discipline were structured at a time when computer technologies were still far from the leading role in education, and therefore, their connection to the educational process was not perceived as something natural and not burdensome. Thirdly, numerical methods are traditionally presented in a separate course of computational mathematics (or course of numerical methods). But A. D. Myshkis noted that in terms of technical university "it is dangerous to allocate all computational issues to a separate section of the mathematics course: such a separation can significantly reduce the idea of algorithmicity in other sections of the course, which appear to be opposed to the calculations and thus blurred in the applied relation"[8]. This view is also relevant for the training of future mathematics and computer science teachers in pedagogical universities. Let's add to this argument another, due to the current state of education: it makes no sense to break the justification of the method and its first trial application. The approximation of the course of functional analysis to computational mathematics contributes to the continuity and coherence of vocational training. Perhaps this is even the only way to fully implement functional analysis in a pedagogical university. The convergence with computational mathematics should be such as to fully prove the theoretical fact to the number: to trace the projection of abstract ideas into the plane of numerical methods and to give an opportunity to immediately test methods in computational practice. Of course, the measure of this convergence should be reasonable, so that functional analysis does not lose its identity and is not substituted by the course of computational mathematics.

To solve these problems, a scientific methodological research was conducted and a set of two textbooks was developed: a summary of lectures[9] and a collection of tasks[10] on functional analysis for pedagogical universities. The basic concepts underlying this project are as follows:

- adaptation of training material to the students' level of preparation and analytical skills;
- cultivation of the applied component of the discipline, which is realized by a combination of functional analysis and computational mathematics;

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[5] Треногин, В. А., Писаревский, В. М., & Соболева, Т. С. (1984). Задачи и упражнения по функциональному анализу. Москва: Наука.

[6] Треногин, В. А. (2007). Функциональный анализ. Москва: Физматлит

[7] Антоневич, А. Б., Ваткина, Е. И., Мазель, М. А., Миротин, А. Р., Мухин, В. В., Радыно, Я. В. et al. (2003). Функциональный анализ и интегральные уравнения: Лабораторный практикум. Минск: БГУ.

[8] Мышкис, А. Д. (2003) О преподавании математики прикладникам. *Математика в высшем образовании, 1,* 37-52.

[9] Бобилєв, Д. Є. (2016). Функціональний аналіз. Кривий Ріг: Dionat.

[10] Бобилєв, Д. Є. (2017). Функціональний аналіз: збірник задач. Кривий Ріг: Dionat.

• modernization of the course for the use of computing means (applied mathematical packages).

Let's consider in more detail how these concepts are implemented in the textbooks[11 12].

The set of lectures[11] contains short theoretical information about the basic modules of functional analysis: the theory of compression operators, the theory of Fourier series in the Hilbert space and the theory of linear operators. Moreover, some complex structures with purely academic values have been disregarded, in particular, those not having a visual application in computational practice.

For example, many elements of the topology and all the adjacent theorems, the concept of conjugate space and operator, the Banach theorem on inverse operators, the theorem on the addition of a metric space (considered at the level of formulation), theorems on the extension of the operator, the functional, and others are excluded.

The notion of a linear operator limitation, a key to most textbooks on functional analysis, is replaced by the concept of continuity which is equivalent to it. The reason for the replacement is that the traditional definition of a limited linear operator does not correspond to the definition of a bounded function taken in the course of mathematical analysis, whereas the universal concept of continuity for an operator in metric spaces is consistent with the continuity of a function from the course of mathematical analysis. The continuity of the operator is defined as the ability to maintain the sequence convergence, since this definition is the easiest and is used in numerical methods.

The course deals only with the description of the Lebesgue integral included for the introduction of the Lebesgue spaces, and is constructed without the support of the theory of measure, since this section of the bachelor's course in mathematical analysis in most pedagogical universities is not taught to future mathematics and computer science teachers, and it cannot be considered in a short course of functional analysis.

The description of the functional analysis is oriented on two basic tasks that are at the intersection of fundamental and applied mathematics: the approximation of functions and the solution of operator equations. In the problems of approximation of functions with orthogonal systems, the questions of accuracy and quality of approximation are raised. When solving operator equations, the issue of convergence of numerical methods and control of the accuracy of the approximate solution are highlighted on the foreground. The problem of uniqueness of the solution, too, does not fall out from the field of consideration. But the problem of existence, rather difficult for comprehension, is set aside and in some places omitted.

In each module of the lecture set, the line of presentation of theoretical material passes from the introduction of basic concepts to the proof of key theorems that have direct access to widely known numerical methods. These "outputs" are usually described in the last paragraphs of the modules. For their compilation, a database of several dozen textbooks in both functional analysis and different sections of computational mathematics was currently analyzed.

The first module is devoted to metric spaces and compression operators. It ends with an overview of the problems in which the application of the principle of compression operators and the method of simple iterations can be used for the approximate solution of equations of different types.

The second module represents the theory of Fourier series in the Hilbert space. Considerable attention is paid to the variety of orthogonal systems: trigonometric, polynomial and systems of step functions. The module completes the description of the Fourier series connection with the problem of approximation and the explanation of such essential features as the character of the convergence of the Fourier series, the specificity of trigonometric and polynomial approximations, and the differences between the Fourier and Taylor series.

The third module is devoted to the theory of linear operators and covers related issues of a functional optimization. The statement ends with a description of the variational and projective approach to the approximate solution of linear operator equations. The method of least squares and the Galerkin method are analyzed in detail. In addition, in the third module, there are other outcomes for computational mathematics, presented in various sections: search for the solution of

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[11] Бобилєв, Д. Є. (2016). Функціональний аналіз. Кривий Ріг: Dionat.

[12] Бобилєв, Д. Є. (2017). Функціональний аналіз: збірник задач. Кривий Ріг: Dionat.

an equation in the form of a Fourier series by its own functions of the operator, solving the integral equation by the method of replacing the nucleus into a degenerate, approximate minimization of the functional by the Ritz method.

In addition to the strictly deductive method of teaching, heuristic methods (using analogy, selective verification) and other learning methods are used, including those based on informal ideas of mathematical constructions. Most paragraphs have a preface outlining the relationship between the proposed mathematical constructs of functional analysis with similar constructs of the mathematical one, geometry or linear algebra, and some of the brightest applications in natural science and technology.

Thus, the course does not claim to reveal the completeness of the theory, but focuses on the algorithmic component of the functional analysis. The presentation of theoretical constructions is simplified to elementary when trying to preserve the classical quality of this discipline and not to miss the idea that is laid down in this section of mathematics. The abstract energy of the functional analysis is substantially curtailed in comparison with the academic course, but is sustained in such a volume that it could reasonably be applied.

The set of lectures and problems are constructed in such a way as to make the subject as accessible as possible for self-study and, from the first lessons, to intensify the student's educational activity in the context of strict curriculum restrictions. If you do not spend lessons on a consistent and detailed presentation of the material, and adapt the study process to the mode of review lectures and tutorials with the problems solving, then it makes possible to optimize the amount of class hours. It is expedient to limit review lectures to a small amount of formal data and to devote a comprehensive discussion of key mathematical ideas of functional analysis and related non-mathematical associations. The course of functional analysis urgently needs this approach, since it brings a certain summary of the accumulated experience in the study of other sections of mathematics.

The review lectures can include, for example, the following issues: the problem of similarity and differences between objects that are numerically solved using different metrics; what is the convergence and the reasons for its diversity; Linearity as a universal positive characteristic of mathematical constructions and nonlinearity as a source of troubles; what is the difference between a schedule and an approximation of functions; the general notion of continuity and the role played by continuity in various numerical methods; relative simplicity and attractiveness of finite-dimensional spaces, numerous methods based on reduction to a finite-dimensional problem. The set of lectures is intended to prepare and inspire the student to reflect on the ideology that lies in the course of functional analysis. Certain relevant review lectures can greatly enhance this effect.

The problem book contains a large bank of multilevel tasks with a large number of options and is designed for a convenient distribution of points in the evaluation: for example, from 1 point for the simplest one-step problem to 10 points for multi-step calculation with implementation in a mathematical package.

Consider examples of problems from the collection.

Exercise 1. Prove directly that if E is a Banach space and M is a closed subspace of E then the quotient space $(E / M, \|\cdot\|_{E / M})$ is complete. [Use Banach's criterion[13]].

Exercise 2. Let M and N be subspaces of the normed space X. Prove that if M is finite dimensional and N is closed then $M + N$ is closed. [Recall that finite-dimensional subspaces of normed spaces are closed; use the quotient map].

Exercise 3. Find a Hilbert space H and a countable family of vectors (x_n), $n \in N$ in H that is summable but not absolutely summable (i.e., $(\|x_n\|)$, $n \in N$ is not summable)[14].

Exercise 4. A sequence in a normed vector space that is convergent is necessarily bounded. Is the same true for nets?

Exercise 5. Prove that the closed unit ball of c_0 has no extreme points.

[13] Lax, P. (2002) Functional analysis. New York John: Wiley & Sons, Inc.
[14] Vincent-Smith G. F. (1991). B4: Analysis. Mathematical Institute notes. Oxford: University of Oxford.

Exercise 6. Let H be a Hilbert space. Prove that every unit vector in H is an extreme point of the closed unit ball H_1. [Note that 1 is an extreme point of F_1]. Deduce that every isometry in $B(H)$ is an extreme point of the closed unit ball $B(H)_1$.

Exercise 7. Let X be a Hausdorff, locally compact space. Prove that $C_0(X)$, the unitization of the algebra of continuous functions on X that vanish at infinity, is topologically isomorphic to $C(X)$, the algebra of continuous functions on X, the one-point compactification of X.

Exercise 8. Let $A = C[z]$ denote the unital algebra of complex polynomials and let $\|p\| := \sup\{|p(\alpha)|: |\alpha| \leq 1\}$ for all $p \in A$. Show that $(A, \|\cdot\|)$ is a unital, normed algebra which is not complete.

Exercise 9. Let A be a (non-unital) Banach algebra such that every element is nilpotent (i. e., for all $a \in A$ there exists $n \in \boldsymbol{N}$ such that $a^n = 0$). Prove that A is uniformly nilpotent: there exists $N \in \boldsymbol{N}$ such that $a^N = 0$ for all $a \in A$. [Consider the decomposition].

Exercise 10. Let A be a unital Banach algebra over **C** and let $e^a := \sum_{n=0}^{\infty} a^n/n!$ for all $a \in A$. Prove that $e^{a+b} = e^a e^b$ if a and b commute. Deduce that e^a is invertible. Prove further that f: $\lambda \rightarrow e^{\lambda a}$ is holomorphic everywhere, with $f'(\lambda) = \mathrm{af}(\lambda) = f(\lambda)a$, for all $a \in A$.

When conducting practical classes on the basis of this teaching-methodical complex, the emphasis is transferred from the manual calculation technique to the organization of the computational process on computers. Such a transfer is inevitable in modern conditions.

Conclusions. The considered educational-methodical complex[15] [16] allows and partially forces to reorganize the educational process of training the discipline "Functional analysis".

It is expedient to use computers and the qualified interpretation of results should become one of the main goals of teaching not only functional analysis, but, mathematics in general at a pedagogical university. The developed problem book[16] makes it possible; on the one hand, to replicate previously acquired skills in solving various problems, on the other hand, it allows students to learn to use mathematical packages. The problem book[16] includes 58 tasks (20 variants of each), all of them have samples or instructions to solutions and a corresponding reference to the set of lectures[15].

Prospects for further research in the field. In the teaching-methodological complex[15] [16] certain specific problems related to the differences of scholastic (educational, theoretical) and computer mathematics are not worked out thoroughly, they require attention at the initial stage of the mathematical packages application. The number of tasks in the problem book[16] showing the typical difficulties that students face when the computer responds in the form of a character expression that may contain special functions, faced by the student for the first time, is insufficient. Applied mathematical packages require a much more responsible attitude to working with data types (numbers, variables, expressions, functions) than it is customary in fast calculations on paper.

[15] Бобилєв, Д. Є. (2016). Функціональний аналіз. Кривий Ріг: Dionat.
[16] Бобилєв, Д. Є. (2017). Функціональний аналіз: збірник задач. Кривий Ріг: Dionat.